An Automatic Adaptive Numerical Method for Lifting Surface Theories

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For problems involving numerical lifting surface calculations, comparisons of the various approximate methods do not always yield satisfactory results. This paper presents a means of determining the accuracy of any approximate method without comparisons. Moreover, it enables the optimum ratio *NS/NC* (number of downwash chords/number of control points in the downwash chord) to be found automatically by just doing the necessary calculations to obtain any prescribed accuracy. For this purpose, a steepest descent method and an appropriate convergence theorem are used. Computations proceed along the direction of $-$grad |grad C_L| in the *NS-NC* diagrams. Although this method is applied here mainly to vortex lattice methods (VLM) for wings in steady flow, it is a universal one applicable to any efficient unsteady VLMs or mode methods.

Nomenclature

AR	= aspect ratio
B_m	= see Eq. (14), assumed to be unity here
b_0	= semispan
$C(y)$	= local semichord containing loading point
$C(\eta)$	= local semichord at lattice end
$C_L, C_{L\alpha}$	= lift coefficient and lift-curve slope (per radian)
C_L'	= approximate limiting value of C_L, through relaxation approach
$C_L^{(m)}$	= C_L obtained at the *m*th step of regular approach
E_m	= error of the *m*th value against the limiting one
E	= specified error
NC	= number of control points in the downwash chord
NS	= number of downwash chords
NT	= $NS \times NC$
$r_m, \bar{r}_m$	= see Eqs. (13) and (18)
ΔS	= $\sqrt{(\Delta X)^2 + (\Delta Y)^2}$
TR	= taper ratio, $C_{\text{tip}}/C_{\text{root}}$
x,y	= Cartesian coordinates in physical plane
$\hat{x},\hat{y}$	= Cartesian coordinates in converted plane
$\epsilon_m, \bar{\epsilon}_m$	= see Eqs. (15) and (19)
Λ	= sweptback angle at midchord

Introduction

UNTIL now various methods have been published concerning numerical calculation of lifting surfaces. Because there are no exact solutions for most of the problems, a question arises in many cases about the accuracy and convergence characteristics.

Recently we introduced the "error-index" parameter,[1-3] which corrects many of the defects in previous methods. The parameter makes it possible to compare a quantity of data in a compact and precise manner. We have tried to extend this parameter to cases where no exact solutions exist. The unavailable exact solution is replaced by a suitable "standard" or "reference" solution that is thought as the best among the available solutions. Thus, the parameter becomes a "relative" error index, not the "true" error index.

In order to overcome this situation, we wanted to develop a scheme that could evaluate the method internally without resorting to comparison with other methods. Other requiremnts were: 1) automatic attainment of the optimum *NS/NC* ratio, and 2) minimum computation time to obtain a prescribed accuracy. We do not know of any published papers describing systematic schemes to achieve these requirements.

In this paper, an efficient scheme for satisfying all of the above requirements is presented. The examples of the application of this scheme are restricted mainly to some vortex lattice methods (VLM) for steady wings in incompressible flow. The results are quite encouraging and suggest more extensive applicable domains of the method. The first step is to consider how to proceed in the *NS-NC* plane in order to achieve the most efficient convergence. The answer is to assume the contours of |grad C_L| in the *NS-NC* plane and to proceed in a direction normal to these contours (a sort of

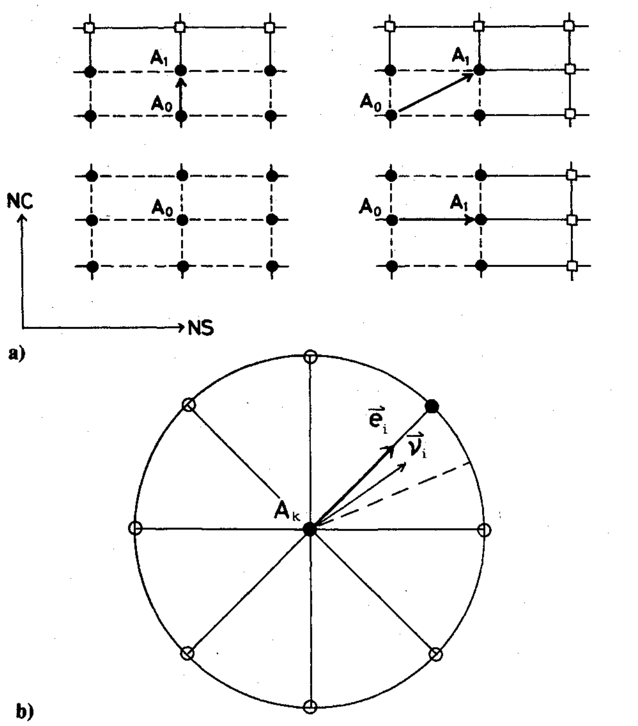

Fig. 1 Descriptions concerning fundamental concept of |grad C_L| scheme: a) nine grid points about a starting point and additional points after proceding one step; b) determination of the direction e_i for proceding one step.

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steepest descent method). This leads to a sequence of C_L values. The second step is to determine the kind of number sequence theorem that should be applied to this sequence of C_L values. For this purpose, we develop a theory described later.

Once the accuracy of a "reference solution" is evaluated absolutely through the present method—i.e., without comparison between any other solutions—we should be able to use the previous "error-index" parameter much more persuasively.

Theoretical Formulation

Optimum Direction in *NS-NC* Plane

Basic Concept

First, suppose that the contours of an equi- $|\mathrm{grad}\ C_L|$ value in the *NS-NC* plane exist, obtained through a method for a lifting surface. It seems that the ultimate limiting value is located in a "basin" of this plane. Thus, the following method is suggested:

Select a suitable starting point. Define the normals to the $|\mathrm{grad}\ C_L|$ contour passing through the starting point. The normals have two directions in which $|\mathrm{grad}\ C_L|$ increases or decreases. Advance by a small step to make both *NC* and *NS* integers in the direction of decreasing $|\mathrm{grad}\ C_L|$. Assume that the point thus reached is the new starting point and repeat the process. Hereafter, this will be referred as the "grad C_L-contour scheme." For most of the cases tested using this scheme, the ultimate limiting value of C_L is located in the basin of the $|\mathrm{grad}\ C_L|$ contours.

Formulation of the Basic Idea

For simplicity, put

$$X = NS/(\Delta NS)_{\min},\quad Y = NC/(\Delta NC)_{\min},\quad Z = C_L = f(X,Y) \tag{1}$$

Thus,

$$\mathrm{grad}\,|\mathrm{grad}\ Z| = \mathrm{grad}\sqrt{f_X^2 + f_Y^2}$$
$$= [\imath(f_X f_{XX} + f_Y + f_{YX}) + \jmath(f_X f_{XY} + f_Y + f_{YY})]/\sqrt{f_X^2 + f_Y^2} \tag{2}$$

where $\imath$ and $\jmath$ denote the unit vectors in the X and Y directions, respectively. Using the centered differences to minimize the truncation errors, the following formulas result:

$$f_X = \frac{1}{2h}\begin{vmatrix} 1 & 0 & -1 \end{vmatrix} \qquad f_{XX} = \frac{1}{h^2}\begin{vmatrix} 1 & -2 & 1 \end{vmatrix}$$

$$f_Y = \frac{1}{2k}\begin{vmatrix} 1 \\ 0 \\ -1 \end{vmatrix} \qquad f_{YY} = \frac{1}{k^2}\begin{vmatrix} 1 \\ -2 \\ 1 \end{vmatrix}$$

$$f_{XY} = f_{YX} = \frac{1}{4hk}\begin{vmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{vmatrix} \tag{3}$$

where $h = \Delta X$ and $k = \Delta Y$. Here, $(\Delta NS)_{\min} = 2$ and $(\Delta NC)_{\min} = 1$, since

$$NS = \begin{matrix} \text{even} & \text{for VLM} \\ \text{odd} & \text{for Lamar's method} \end{matrix} \tag{4}$$

The procedure for proceding in the X-Y diagram is:

1) Select a suitable starting point, $A_0(X_0, Y_0)$.

2) Put $X_0 = X_i$ and $Y_0 = Y_i$. Obtain values of f at the nine points shown in Fig. 1a. Calculate the necessary derivatives using Eqs. (3).

3) Calculate

$$|\mathrm{grad}\ Z|_i = (\sqrt{f_X^2 + f_Y^2})_i \tag{5}$$

and then ν_i, the unit vector in the direction $-\mathrm{grad}\,|\mathrm{grad}\ Z|$.

4) Stop the calculation if it is found that the point (X_i, Y_i) is located at a satisfactory condition according to the convergence theorem described below.

5) Otherwise, go to point A_1: divide all of the directions about point A_0 into eight equal sectors, one of which will contain the direction of ν_i. Define e_i, counting fractions of more than a half inclusive as one and eliminating those of less than a half. See Fig. 1b. Define point A_1 in the direction of e_i. Thus, the direction error lies within ± 22.5 deg.

6) If the direction of e_i coincides with X axis,

$$\Delta X = 1,\quad \Delta Y = 0,\ \text{and}\ \Delta S = 1 \tag{6}$$

If the direction of e_i coincides with Y axis,

$$\Delta X = 1,\quad \Delta Y = 1,\ \text{and}\ \Delta S = 1 \tag{7}$$

Otherwise,

$$\Delta X = 1,\quad \Delta Y = 1,\ \text{and}\ \Delta S = \sqrt{2} \tag{8}$$

7) Repeat steps 2-6 taking a new starting point at A_1. The □ marks in Fig. 1a denote the points where additional values of Z should be calculated.

Suitable Convergence Theorem

Let $S = 0$ at $A_0(X_0, Y_0)$ and $Z_0, Z_1, Z_2, \ldots,$ correspond to $S_0 = 0$, S_1, $S_2, \ldots,$ respectively. Then the limiting value of the number sequence Z_i is written as

$$Z = Z_\infty = Z_m + \sum_{k=m+1}^{\infty} \Delta Z_k,\quad \Delta Z_k = Z_k - Z_{k-1} \tag{9}$$

where ΔZ_k is caused by $\Delta S_k = S_k - S_{k-1}$. ΔS_k need not always be uniform. Assume that all ΔZ_k have a constant sign, positive or negative, then

$$\pm \sum_{k=m+1}^{\infty} \Delta Z_k = \pm \sum_{k=m+1}^{\infty} \frac{\Delta Z_k}{\Delta S_k}\Delta S_k \le \sum_{k=m+1}^{\infty} |\mathrm{grad}\ Z|_k \Delta S_k \tag{10}$$

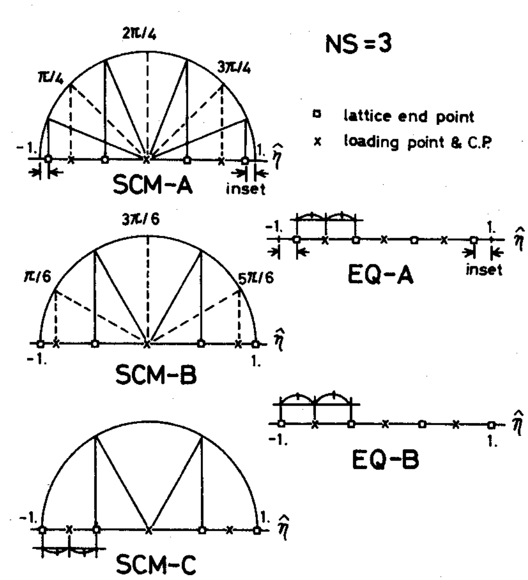

Fig. 2 Spanwise division for five VLMs.

where the double sign $\pm$ corresponds to $\Delta Z \gtrless 0$. Hereafter, in all number sequences, every increment ΔZ_k must have a constant sign. In our experience, approximate numerical methods are generally poor if all of the increments do not have a constant sign. Therefore, this restriction is not critical. Let us introduce functions $g_u(S)$ and $g_\ell(S)$ such that

$$g_\ell(S) \leq |\text{grad } Z|_k \leq g_u(S), \quad k \geq m+1 \tag{11}$$

If the integral

$$\int_{S_m}^{\infty} g_u(S)\,\mathrm{d}S$$

converges, then Eq. (10) also does so.

Now we restrict our interest to the case (an "N^{-r} type series")

$$g_u(N) \sim N^{-r}, \quad r > 1 \tag{12}$$

based on our preliminary investigations.

"Regular Approach"

Assume

$$g_u(S) \leq A_m / S^{r_m}, \quad r_m > 1, \quad S \geq S_m \tag{13}$$

and

$$B_m \left| \frac{\Delta Z}{\Delta S} \right|_{m+1} = g_u(S_m) = \frac{A_m}{S_m^{r_m}}, \quad S = S_m \tag{14}$$

where B_m is to be a suitable constant larger than unity. Then

$$\sum_{k=m+1}^{\infty} \left| \frac{\Delta Z}{\Delta S} \right|_k \Delta S_k \leq \int_{S_m}^{\infty} g_u(S)\,\mathrm{d}S = \frac{A_m}{r_m - 1} S_m^{-r_m+1}$$

$$= \frac{B_m S_m}{r_m - 1} \left| \frac{\Delta Z}{\Delta S} \right|_{m+1} = \epsilon_m \tag{15}$$

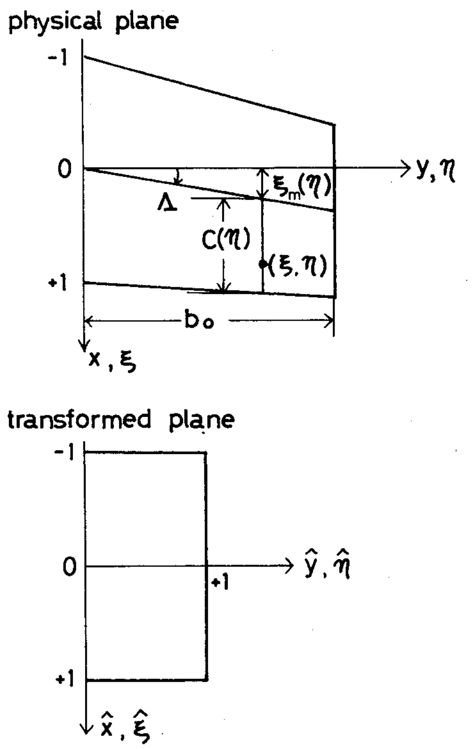

Fig. 3 Cartesian coordinates in the physical and converted planes.

Thus

$$E_m = \pm (Z - Z_m) \leq \epsilon_m \tag{16}$$

The parameter r_m is obtained through

$$r_m = \log\left[\left| \frac{\Delta Z}{\Delta S} \right|_{m+1} \Big/ \left| \frac{\Delta Z}{\Delta S} \right|_m \right] \Big/ \log\left(\frac{S_m}{S_{m+1}} \right) \tag{17}$$

Our numerical experiments show that r_m often fluctuates with m variation. This fluctuation deteriorates the numerical work. A counterplan is to introduce an averaged value of r_m and the corresponding ϵ_m,

$$\bar{r}_m = (r_1 + r_2 + r_3 + \ldots + r_m)/m \tag{18}$$

$$\bar{\epsilon}_m = \frac{S_m}{\bar{r}_m - 1} \left| \frac{\Delta Z}{\Delta S} \right|_{m+1} \tag{19}$$

then

$$E_m \leq \bar{\epsilon}_m \tag{20}$$

This device improves things remarkably. The "regular approach" is as follows: 1) specify E as the allowable error of the number sequence $\{Z_i\}$ from the ultimate value Z_∞, 2) repeat the computation until $\bar{\epsilon}_m \leq E$, and 3) then Z_m may be adopted as an approximate value of Z_∞.

"Relaxation Approach"

This method considerably eases the numerical work needed to obtain the approximate ultimate value, although the accuracy cannot be evaluated quantitatively. Equation (16) suggests that it is reasonable to define

$$Z' = Z_m \pm \bar{\epsilon}_m \quad \text{or} \quad C_L' = C_L^{(m)} \pm \bar{\epsilon}_m \tag{21}$$

Then Z' may be much closer to the ultimate value than Z_m. But the error $|Z - Z'|$ is not theoretically known.

Applications

Description of Lifting Surface Methods

Although this scheme may have many applications, we will show only a few in this paper. The selected cases are five VLMs[6] and one mode method, all of which concern wings in steady incompressible flow. The planforms of the wings are tapered and swept back.

The chordwise layout of the control and loading points are common to all VLMs, but the spanwise layouts are different. Let $\hat{x}_s$ and $\hat{\xi}_j$ be the chordwise control and loading points in the converted plane, respectively. Then

$$\hat{x}_s = -\cos[2s\pi/(2NC+1)]$$

$$\hat{\xi}_j = -\cos[(2j-1)\pi/(2NC+1)] \tag{22}$$

The spanwise layout contains three semicircular and two equispaced divisions. Let $\hat{y}_r$ and $\hat{\eta}_n$ be the spanwise control points and lattice ends, respectively, where $1 \leq r \leq NS$ and $1 \leq n \leq NS+1$. Then

SCM-A:

$$\hat{y}_r = -\cos[r\pi/(NS+1)]$$

$$\hat{\eta}_n = -\cos[(2n-1)\pi/2(NS+1)] \tag{23a}$$

SCM-B:

$$\hat{y}_r = -\cos[(2r-1)\pi/2NS]$$

$$\hat{\eta}_n = -\cos[(n-1)\pi/NS] \tag{23b}$$

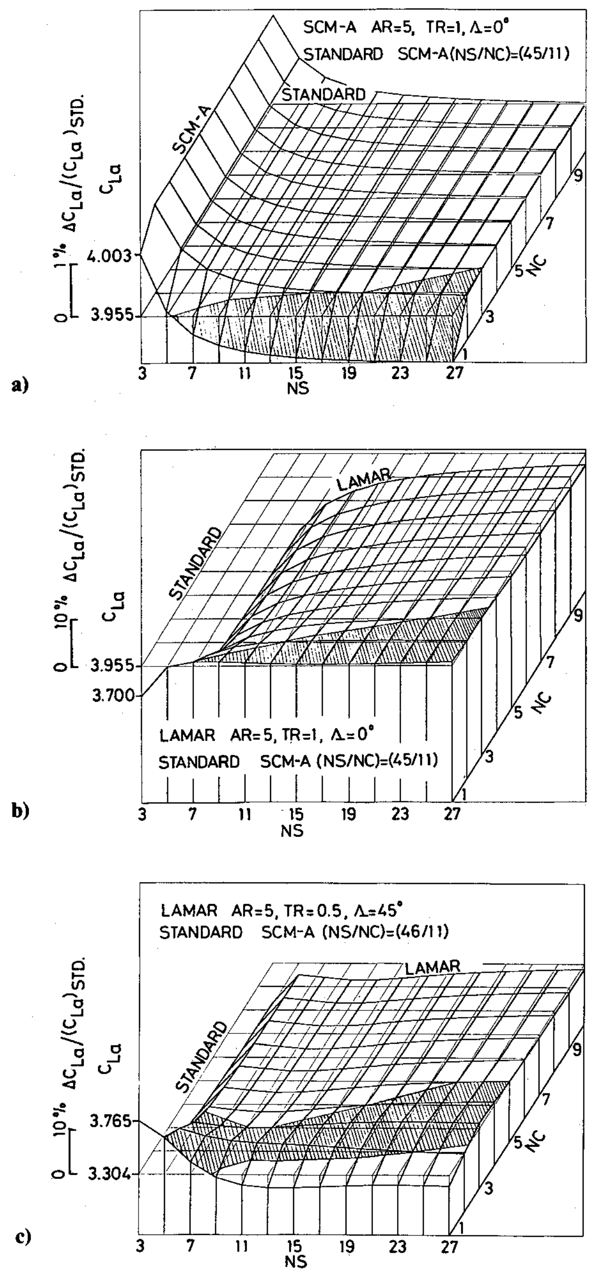

Fig. 4 $C_{L\alpha}$ **in** *NS-NC* **diagram: a) SCM-A, rectangular wing of** $AR=5$ **(shaded part means SCM-A ≤ standard); b) Lamar's method, rectangular wing of** $AR=5$ **(shaded part means Lamar ≥ standard); c) Lamar's sweptback tapered wing of** $AR=5$**,** $TR=0.5$**, and** $\Lambda=45$ **deg (shaded part means Lamar ≥ standard).**

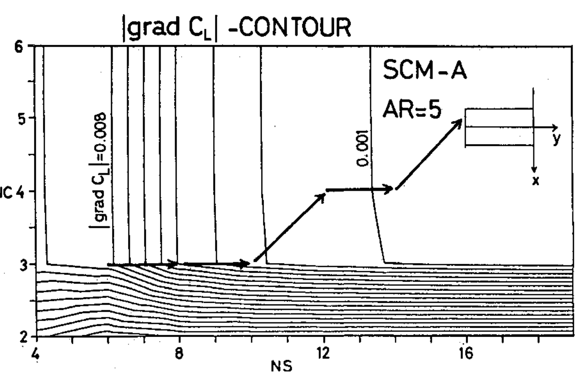

Fig. 5 Optimum procedure through SCM-A for rectangular wing of $AR=5$**.**

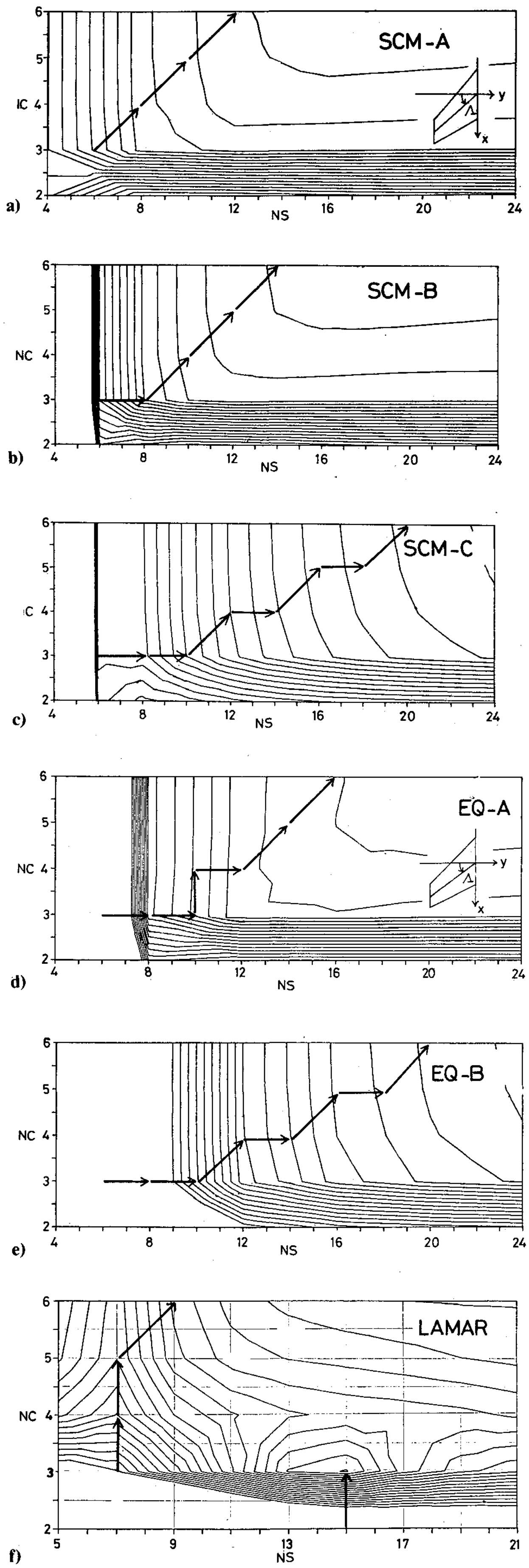

Fig. 6 Optimum procedure for sweptback wing of $AR=2$**,** $TR=0.5$**, and** $\Lambda=45$ **deg: a) SCM-A; b) SCM-B; c) SCM-C; d) EQ-A; e) EQ-B; f) Lamar's.**

SCM-C:

$$\hat{y}_r = -(1/2)\{\cos[(r-1)\pi/NS] + \cos[r\pi/NS]\}$$

$$\hat{\eta}_n = -\cos[(n-1)\pi/NS] \tag{23c}$$

EQ-A:

$$\hat{y}_r = -1 + S + (1-S)(2r-1)/NS$$

$$\hat{\eta}_n = -1 + S + 2(1-S)(n-1)/NS \tag{23d}$$

$$S = 1/(2NS)$$

EQ-B:

$$\hat{y}_r = -1 + (2r-1)/NS$$

$$\hat{\eta}_n = -1 + 2(n-1)/NS \tag{23e}$$

The spanwise divisions are shown in Fig. 2.

The relations of the coordinates between the physical and converted planes (Fig. 3) are written as

$$x = (1 - \lambda_0 |\hat{y}|)\hat{x} + A_0 b_0 |\hat{y}|$$

$$\xi = (1 - \lambda_0 |\hat{\eta}|)\hat{\xi} + A_0 b_0 |\hat{\eta}|$$

$$y = b_0 \hat{y}, \qquad \eta = b_0 \hat{\eta} \tag{24}$$

where

$$\lambda_0 = 1 - TR, \quad A_0 = \tan\Lambda, \quad b_0 = AR(1+TR)/2 \tag{25}$$

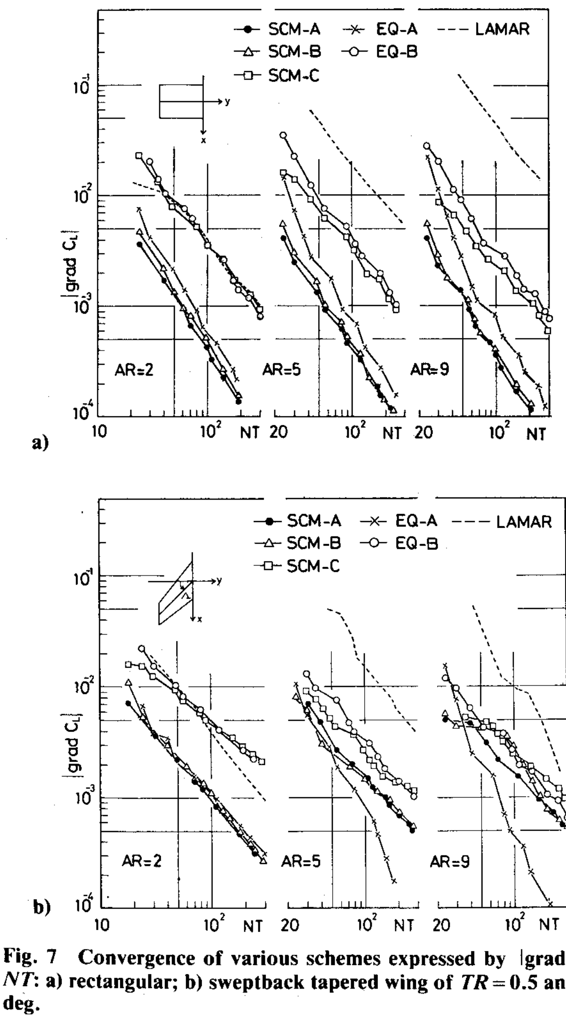

Fig. 7 Convergence of various schemes expressed by $|\mathrm{grad}\ C_L|$ vs NT: a) rectangular; b) sweptback tapered wing of $TR = 0.5$ and $\Lambda = 45$ deg.

It is noteworthy to make a brief comment about the "tip inset" contained in EQ-A [Eq. (23d)]. References 7 and 8 point out that the optimum tip inset is 25% of the most outside lattice span. The tip inset of the SCM-A is not optimum, since it is too small.

Only one mode method, Lamar's scheme[4] is investigated. Preliminary calculations show that Lamar's method yields almost the same results as the Davies' method[5] for the steady case.

Results and Discussions

Figure 4 shows $C_{L\alpha}$ in *NS-NC* diagrams of the SCM-A and Lamar's schemes. Rectangular wings produce monotonically curved surfaces for both schemes, except in some domains where $NC < 2$ or $NS < 7$. But readers should note the difference in the ordinate scalings between the two schemes. For a swept tapered wing only Lamar's scheme is presented, see Fig. 4c. It exhibits a strange complicated curved surface. These figures suggest that Lamar's scheme has poor convergence characteristics and that it is difficult to find a suitable convergence theorem for this problem.

Figure 5 shows an application of the present method for SCM-A to a rectangular wing. It appears that the present method works satisfactorily. Similar diagrams for other schemes are omitted for reasons of space. Figure 6a shows the results for a tapered sweptback wing with $AR = 2$, $\Lambda = 45$ deg, and $TR = 0.5$. The five VLM solutions are reasonable, while Lamar's shows a strange pattern. The two arrowed lines tend to different values. Of course, the end of the shorter line is incorrect, since it terminates at a false hill of C_L (see Fig. 4c).

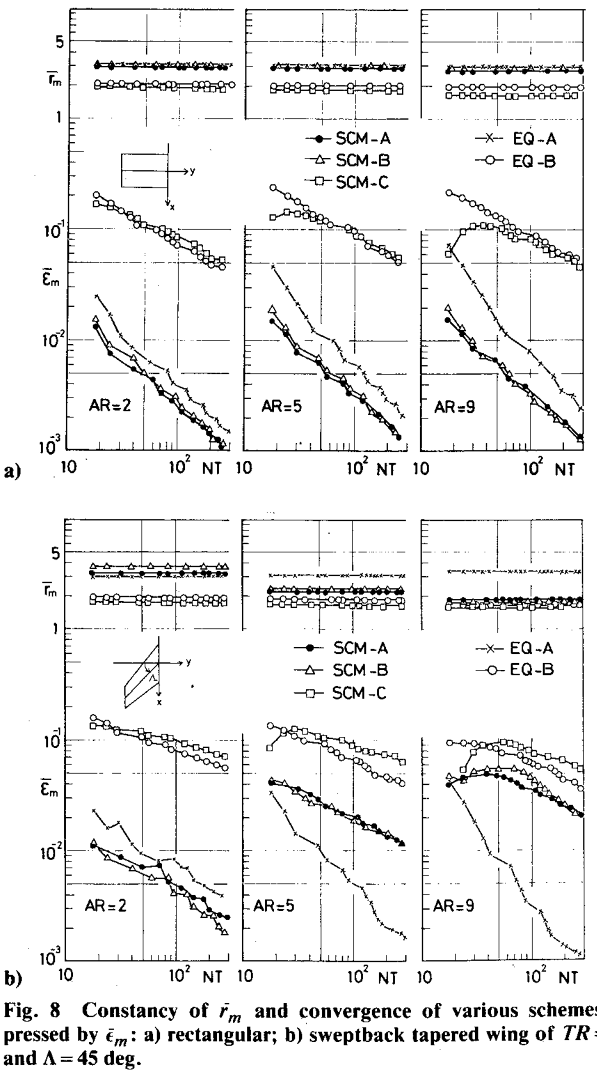

Fig. 8 Constancy of $\bar{r}_m$ and convergence of various schemes expressed by $\bar{\epsilon}_m$: a) rectangular; b) sweptback tapered wing of $TR = 0.5$ and $\Lambda = 45$ deg.

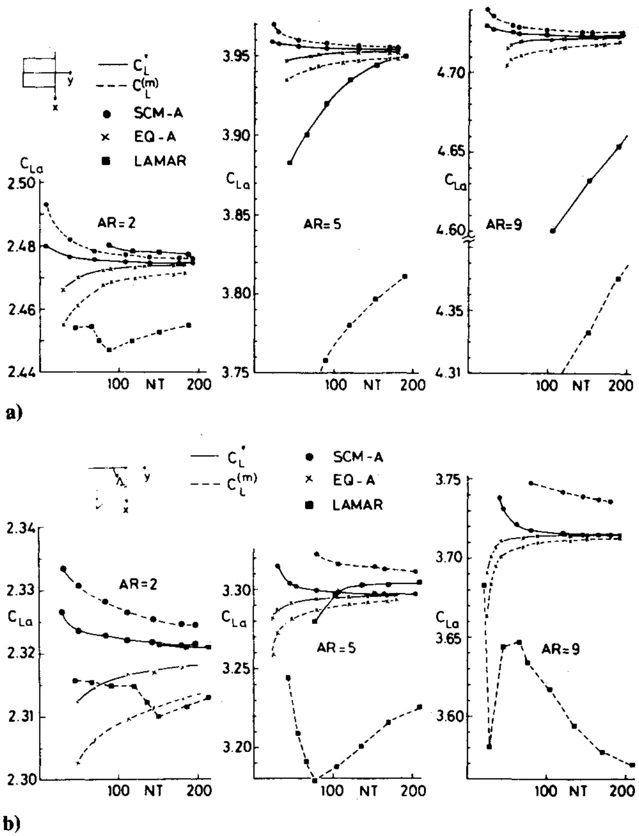

Fig. 9 Comparison of relaxation and regular approaches: a) rectangular; b) sweptback tapered wing of $TR = 0.5$ and $\Lambda = 45$ deg.

A word of caution, however: the equi- |grad C_L | curves of these figures are shown for demonstration purposes only and may be unnecessary for practical applications of the present method.

Figure 7 shows |grad C_L| vs NT as obtained with the present method. For rectangular wings, the SCM-A scheme is the best, while Lamar's is the worst. But, for swept tapered wings, EQ-A is best, at especially higher aspect ratios. The linearity of the curves is fairly good. Figure 8 shows $\bar{\epsilon}_m$ and r_m vs NT. The values of r_m are larger than unity, nearly equal to 3 for the best group and 2 for the worst group. Averaging r_m as shown in Eq. (18) supresses the fluctuations quite well. The linearity of the $\bar{\epsilon}_m$ vs NT curves is also good for rectangular wings and fairly good for swept tapered wings. Note that these good linearities result from averaging r_m. The remaining nonlinearities seem to be responsbile for the fact that the direction $\boldsymbol{\nu}_i$ is replaced by $\boldsymbol{e}_i$ (see Fig. 1b). We find again that the EQ-A scheme is approximately best irrespective of AR and Λ.

Figure 9 compares C_L' and $C_L^{(m)}$ for three schemes and various parameters. It can be seen that C_L' (solid curves) converges much more rapidly than $C_L^{(m)}$ (broken curves). The convergence of Lamar's method is poor, especially for higher aspect ratios. These figures suggest that the "relaxation approach" is superior to the "regular approach" in every scheme.

Conclusions

Important problems in numerical methods for lifting surfaces are to realize the optimum NS/NC ratio, to evaluate the error without comparison with other methods, and to stop the computations automatically when a specified accuracy is reached. The presented "automatic adaptive numerical method" is quite encouraging. Although the method is applied here mainly to the vortex lattice method for wings in steady flow, it is a universal one in prinicple. Several essential techniques are developed, for example, averaging the parameter r_m, a sort of relaxation technique, and the steepest descent technique for C_L in the NS-NC diagram. Proof that the present method works satisfactorily for general compressible unsteady cases should be made in future investigations.

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